Student Number:
Teacher:

St George Girls High School

Mathematics Extension1

2021 Trial HSC Examination

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen.
- Calculators approved by NESA may be used.
- A reference sheet is provided.
- For questions in **Section I**, use the multiple-choice answer sheet provided
- For questions in **Section II**:
  - Answer the questions in the writing booklets provided
  - Extra writing booklets are provided if needed
  - Start each question in a new writing booklet
  - Show relevant mathematical reasoning and/or calculations
  - Marks may not be awarded for incomplete or poorly presented solutions, or where multiple solutions are provided

Total marks: 70

Section I – 10 marks (pages 2 – 8)

- Attempt Questions 1–10.
- Allow about 15 minutes for this section

Section II – 60 marks (pages 9 – 15)

- Attempt Questions 11–15.
- Allow about 1 hour and 45 minutes for this section

Q1-10	/10
Q11	/12
Q12	/12
Q13	/12
Q14	/12
Q15	/12
TOTAL	/70
	%

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Section I-Multiple Choice

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section.
Use the multiple-choice answer sheet provided for Questions 1 - 10.

1. What is the domain of the function $f(x) = 2\cos^{-1} 2x$?

A. $[-1, 1]$.

B. $[-2, 2]$

C. $[0, \pi]$.

D. $\left[-\frac{1}{2}, \frac{1}{2}\right]$

2. Which of the following is equal to $\sin x + \sqrt{3}\cos x$?

A. $2\sin\left(x + \frac{\pi}{3}\right)$

B. $2\sin\left(x + \frac{\pi}{6}\right)$

C. $2\sin\left(x - \frac{\pi}{3}\right)$

D. $2\sin\left(x - \frac{\pi}{6}\right)$

Section A continued.

3. A piece of hot metal is placed in a room with a surrounding air temperature of 15˚C and allowed to cool. It loses heat according to Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - A)$$

where T is the temperature of the metal in degrees Celsius at time t minutes, A is the surrounding air temperature and k is a positive constant.
After 5 minutes the temperature of the metal is 75˚C, and after a further 3 minutes it is 45˚C. What is the value of k in the above equation?

A. $3 \ln 0.5$

B. $3 \ln 2$

C. $\dfrac{\ln 0.5}{3}$

D. $\dfrac{\ln 2}{3}$

4. A circle is represented by the parametric equations

$$x = 2 + 4\cos\theta$$
$$y = 3 + 4\sin\theta$$

What are the centre and radius of the circle?

A. Centre, $(-2, -3)$ radius 4.

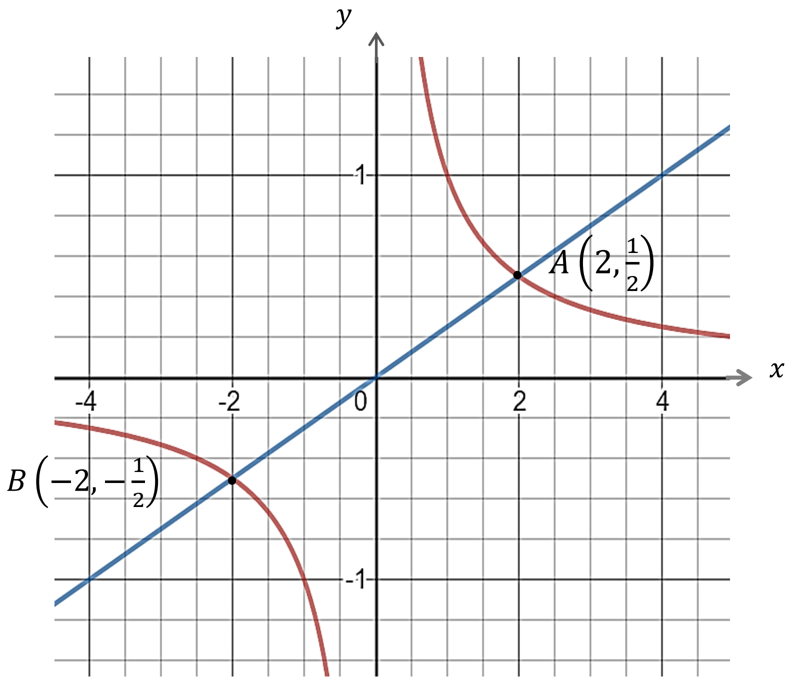
B. Centre$(-2, -3)$, radius 16.

C. Centre $(2, 3)$, radius 4.

D. Centre$(2, 3)$, radius 16.

Section A continued

5. The functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{4}$ intersect at the points $A\left(2, \frac{1}{2}\right)$ and $B\left(-2, -\frac{1}{2}\right)$ as shown in the graph below.



Using the graph, or otherwise, find the solution of the inequation

$$\frac{1}{x} > \frac{x}{4}$$

A. $x < -2\,, x > 2$

B. $-2 < x < 0\,, x > 2$

C. $x < -2\,, 0 < x < 2$

D. $-2 < x < 0\,, 0 < x < 2$

Section A continued

6. The points P, Q and R lie on a straight line.

The vector $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and the vector $\overrightarrow{PR} = 2\overrightarrow{PQ}$. The coordinates of P are $(2, -2)$.

What are the coordinates of R?

A. $(-2, 4)$.

B. $(-6, 8)$

C. $(-6, 10)$

D. $(6, -10)$

7. Kylie drives her car from the origin all along the vector $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$. Then from A she drives all along the vector $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ and after that, from B she drove to C along the vector $\overrightarrow{BC} = \begin{pmatrix} p \\ p+2 \end{pmatrix}$, where p is a positive constant.

Given that the vector position $\overrightarrow{OC}$ from her starting to her final position has a magnitude of 13, what is the value of p?

A. 5

B. 2

C. 3

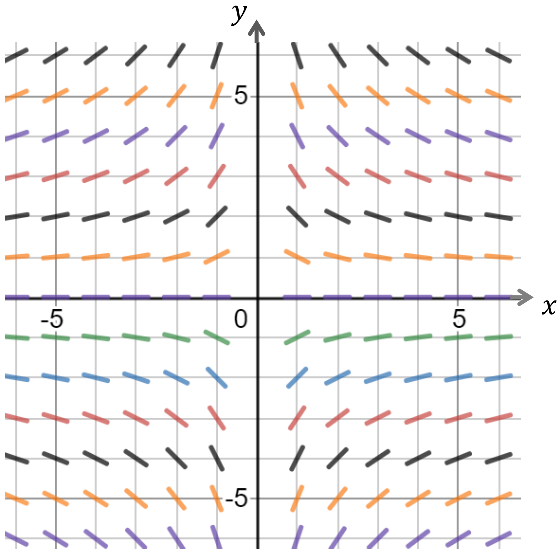
D. 1

Section A continued

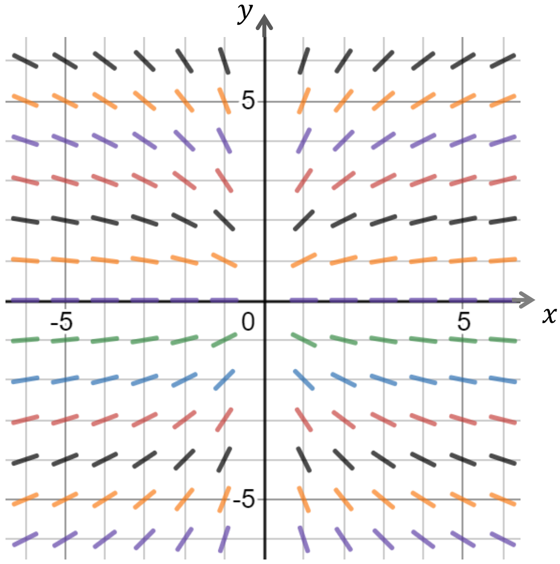
8. Which of the following best represents the direction field for the differential equation

$\frac{dy}{dx} = -\frac{y}{2x}$?

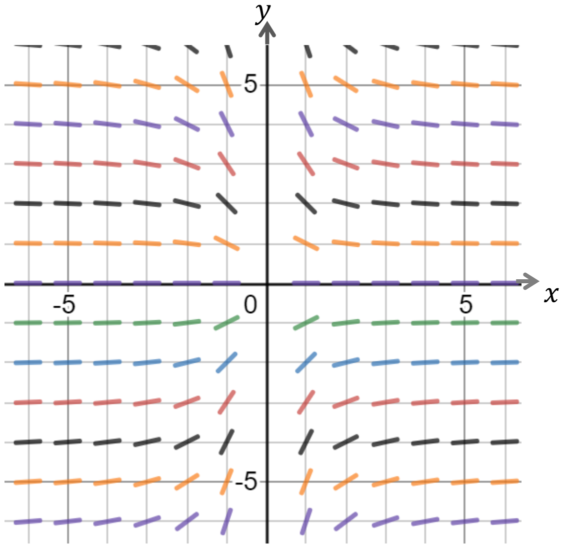
A.



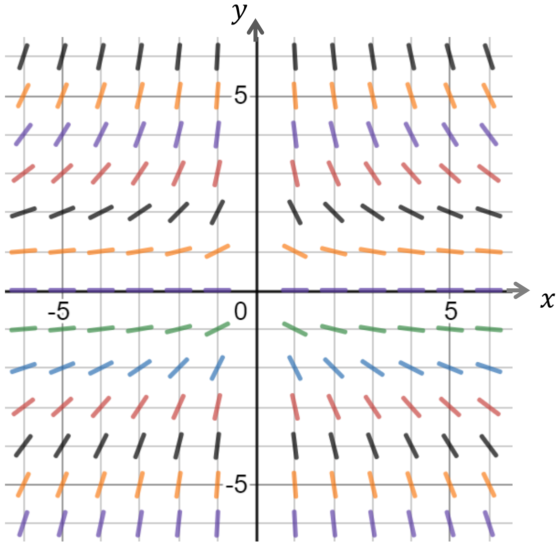
B.



C.



D.



Section A continued

9. Gina has 11 jars of paint, each containing a different coloured paint. One of these jars contains sky-blue paint colour.
She is to select a group of five of these paints and then arrange to colour the five stars shown below. Each star is to be painted in a different colour.

☆ ☆ ☆ ☆ ☆

The sky-blue colour is to be included in any group that Gina selects, and it can be used to paint any star except the star in the centre.

In how many different ways can she paint these five stars?

A. $4 \times {}^{11}P_5$

B. $4 \times {}^{10}P_4$

C. $5 \times {}^{11}P_5$

D. $5 \times {}^{10}P_4$

10. What is the value of k such that $\int_0^k \frac{1}{\sqrt{4-9x^2}}\ dx = \frac{\pi}{18}$?

A. $-\frac{1}{3}$

B. $\frac{\sqrt{3}}{2}$

C. $\frac{\sqrt{3}}{6}$

D. $\frac{1}{3}$

Section II
60 marks
Attempt Questions 11 – 15
Allow about 1 hour and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available. In Questions 11-15, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (12 marks) Start a NEW Writing Booklet. **Marks**

(a) Given that $(x-3)$and $(x+2)$ are factors of $f(x) = x^3 + x^2 + px + q$.

(i) Find the values of p and q. **2**

(ii) Factorise $f(x)$ completely. **2**

(b) For what values of x is $\frac{x}{x-1} \geq -2$? **3**

(c) Use the principle of mathematical induction to show that $9^n + 7$ is divisible by 8 for all integers $n \geq 1$. **3**

(d) Use the pigeonhole principle to determine how many integers must be selected from the numbers 5, 6, 7, 8, 9 and 10 so that at least 2 of the numbers will have a difference of 3? **2**

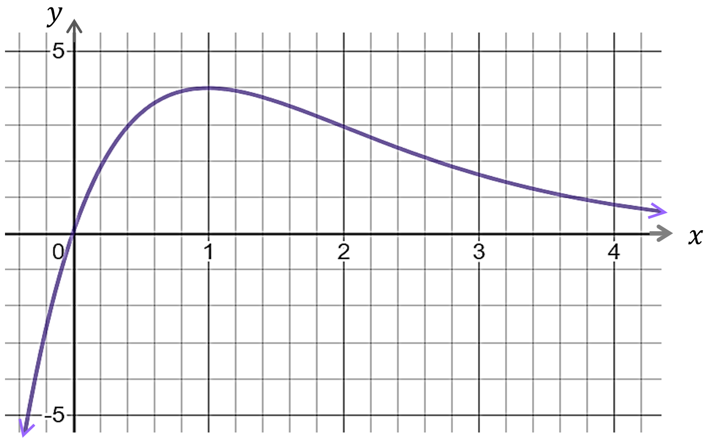
Question 12 (12 marks) Start a NEW Writing Booklet. **Marks**

(a) Solve the differential equation $\frac{dy}{dx} = \frac{-x}{1+y^2}$, given that $y(-1) = 1$.

Express your answer in the form $ay^3 + bx^2 + cy + d = 0$, where a, b, c and d are integers. **3**

(b) Use the substitution $u = 3 + e^x$ to find the exact value of $\int_0^{\ln 6} \frac{e^x}{\sqrt{3+e^x}}\,dx$. **3**

(c)



Sketch the graphs of:

(i) $y = f(|x|)$. **2**

(ii) $y^2 = f(x)$. **2**

(Use the graphs provided at the end of this booklet).

(d) Find the general solution of the differential equation $\frac{dy}{dx} = \frac{4\,x^3}{\sin^2 y}$.

You may leave your answer in terms of y, $\sin 2y$ and x^4. **2**

Question 13 (12 marks) Start a NEW Writing Booklet. **Marks**

(a) Let $f\,(x) = \cos^{-1}\dfrac{1}{x} + \operatorname{cosec}^{-1} x$ for $x \geq 1$.

(i) Prove that $\operatorname{cosec}^{-1} x = \sin^{-1}\dfrac{1}{x}$ **1**

(ii) Hence, show that $f'(x) = 0$. **2**

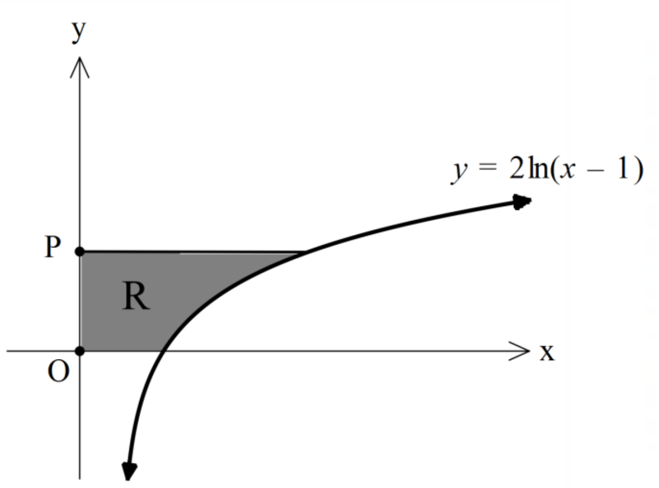
(iii) Prove the identity: $\cos^{-1}\dfrac{1}{x} + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}$ **1**

(b) Consider the points $P\,(\,a, 2a\,), Q\,(\,-\,a, 5a\,), R\,(\,3a, 4a\,)$ and $S\,(\,9a, 12a\,)$, where a is a positive real number.

Given the length of the projection of $\overrightarrow{PQ}$ onto $\overrightarrow{RS}$ is 12, find the value of a. **3**

Question 13 continued

(c) The diagram below shows the curve with equation $y = 2\ln(x-1)$.



The point P has coordinates $(0, p)$.The shaded region R is bounded by the curve and the lines $x = 0$, $y = 0$ and $y = p$. The units of the axes are centimetres.

(i) The region R is rotated completely about the y-axis to form a solid.

Show that the volume, $V\ cm^3$ of the solid is given by:

$$V = \pi\left(e^p + 4e^{\frac{1}{2}p} + p - 5\right).$$ **3**

(ii) It is given that the point P is moving in the positive direction along the y-axis at a constant rate of 0.2 cm/min.

Find the rate at which the volume of the solid is increasing at the instant when $p = 4$, giving your answer correct to 2 significant figures. **2**

Question 14 (12 marks) Start a NEW Writing Booklet. **Marks**

(a) An area of $A\ cm^2$ is occupied by a bacterial colony. The colony has its growth modelled by the logistic equation $\frac{dA}{dt} = \frac{1}{25}\ A(50 - A)$,where $t \geq 0$ and t is measured in days. At time $t = 0$, the area occupied by the bacterial colony is $\frac{1}{2}\text{cm}^2$.

(i) Show that $\frac{1}{A(50-A)} = \frac{1}{50}\left(\frac{1}{A} + \frac{1}{50-A}\right)$. **1**

(ii) Hence, show that the solution to the logistic equation is $A = \frac{50}{1+99e^{-2t}}$. **3**

(iii) According to this model, what is the limiting area of the bacterial colony? **1**

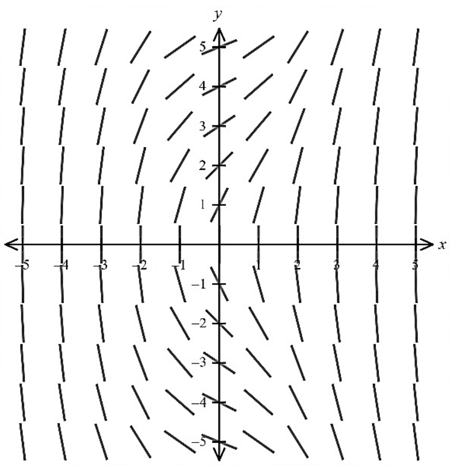
(b) Find $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x\ dx$. **2**

(c) (i) Differentiate $y = x\cos^{-1}x$,with respect to x. **1**

(ii) Hence, find $\int \cos^{-1}x\ dx$.
You **may** use the substitution $u = 1 - x^2$. **2**

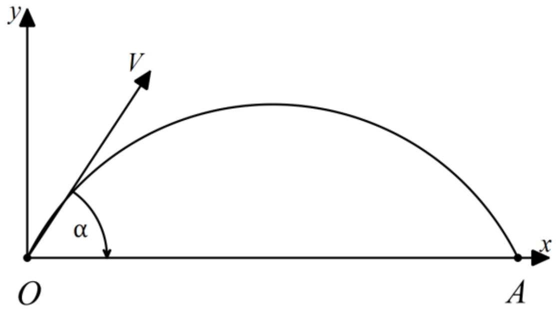
(d) A direction field diagram for the differential equation $\frac{dy}{dx} = \frac{3x^2+4}{2y}$ is shown below. Use it to sketch two possible solution curves, one through $(-1, 0)$ and the other through $(2, 0)$. **2**

Use the diagram at the end of this booklet.



Question 15 (12 marks) Start a NEW Writing Booklet. **Marks**

(a) From a point O on horizontal ground, a particle P is projected under gravity with speed V in a direction making an angle α with the horizontal.
The particle strikes the ground again at the point A.



The equations of motion of P are $x = Vt\cos\alpha$ and $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$.
(Do **NOT** prove these).

(i) Find an expression for the time of flight. **1**

(ii) Show that the distance $OA = \dfrac{V^2\sin 2\alpha}{g}$. **1**

At the instant when P is at the highest point of its path, a second particle Q is projected from O with speed U in a direction making an angle β with the horizontal.The particle Q strikes the ground at A at the same instant as P.

(iii) Given that $\tan\alpha = \frac{4}{3}$, show that $U = \dfrac{2\sqrt{10}\,V}{5}$. **4**

(b) (i) By using $\cos 4\theta = 2\cos^2 2\theta - 1$, show that:

$$8\cos^4\theta - 8\cos^2\theta + 1 - \cos 4\theta = 0.$$ **2**

(ii) Using part (i) and by letting $x = \cos\theta$ into the equation

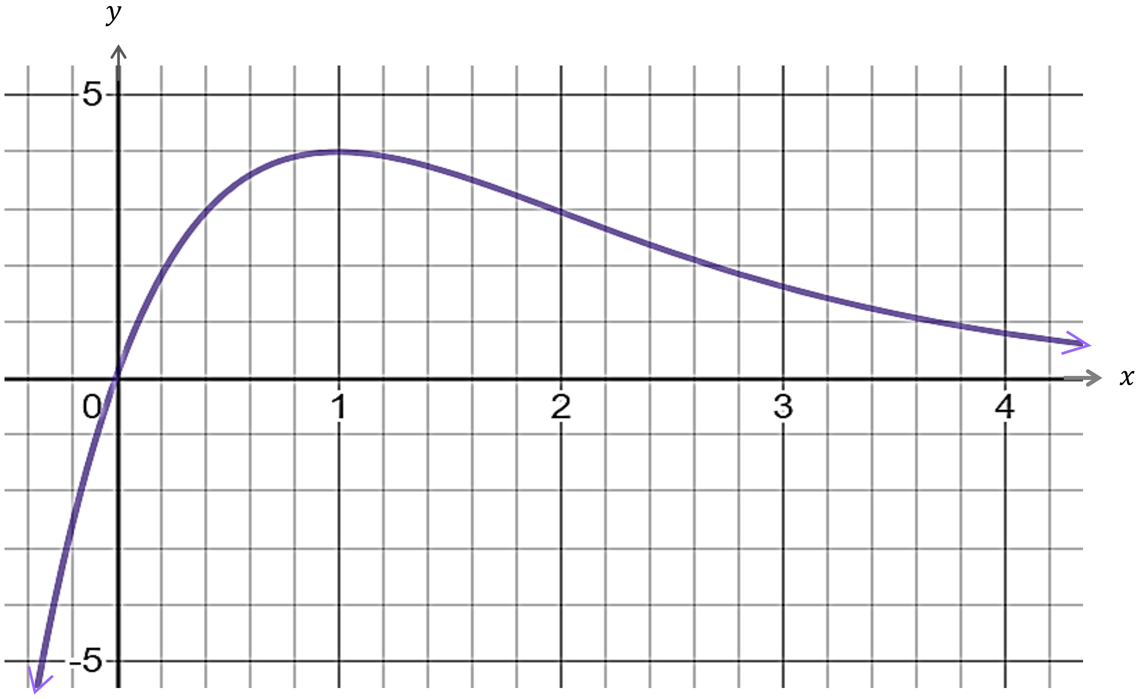
$16x^4 - 16x^2 + 2 - \sqrt{2} = 0$, show that $\cos 4\theta = \dfrac{\sqrt{2}}{2}$. **2**

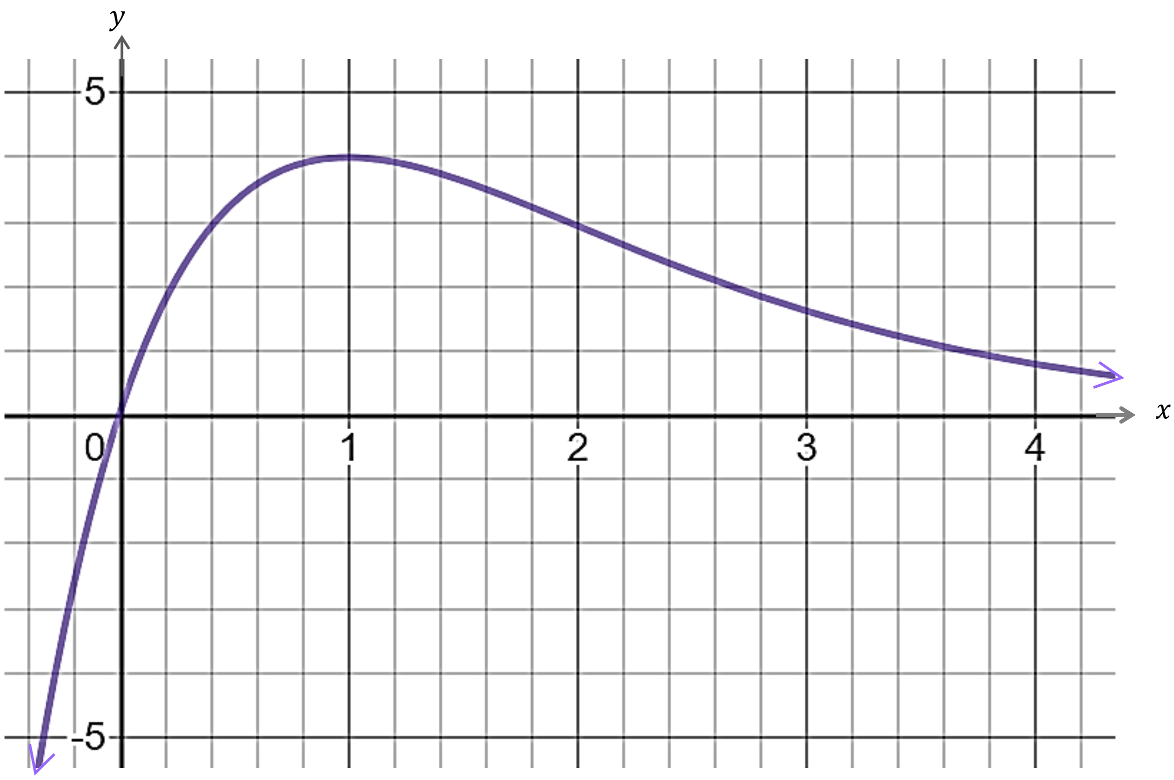
(iii) Hence, prove that $\cos^2\frac{\pi}{16}\cos^2\frac{7\pi}{16} = \dfrac{2-\sqrt{2}}{16}$. **2**

THE END OF EXAMINATION

Name:____

Teacher:______________________________________





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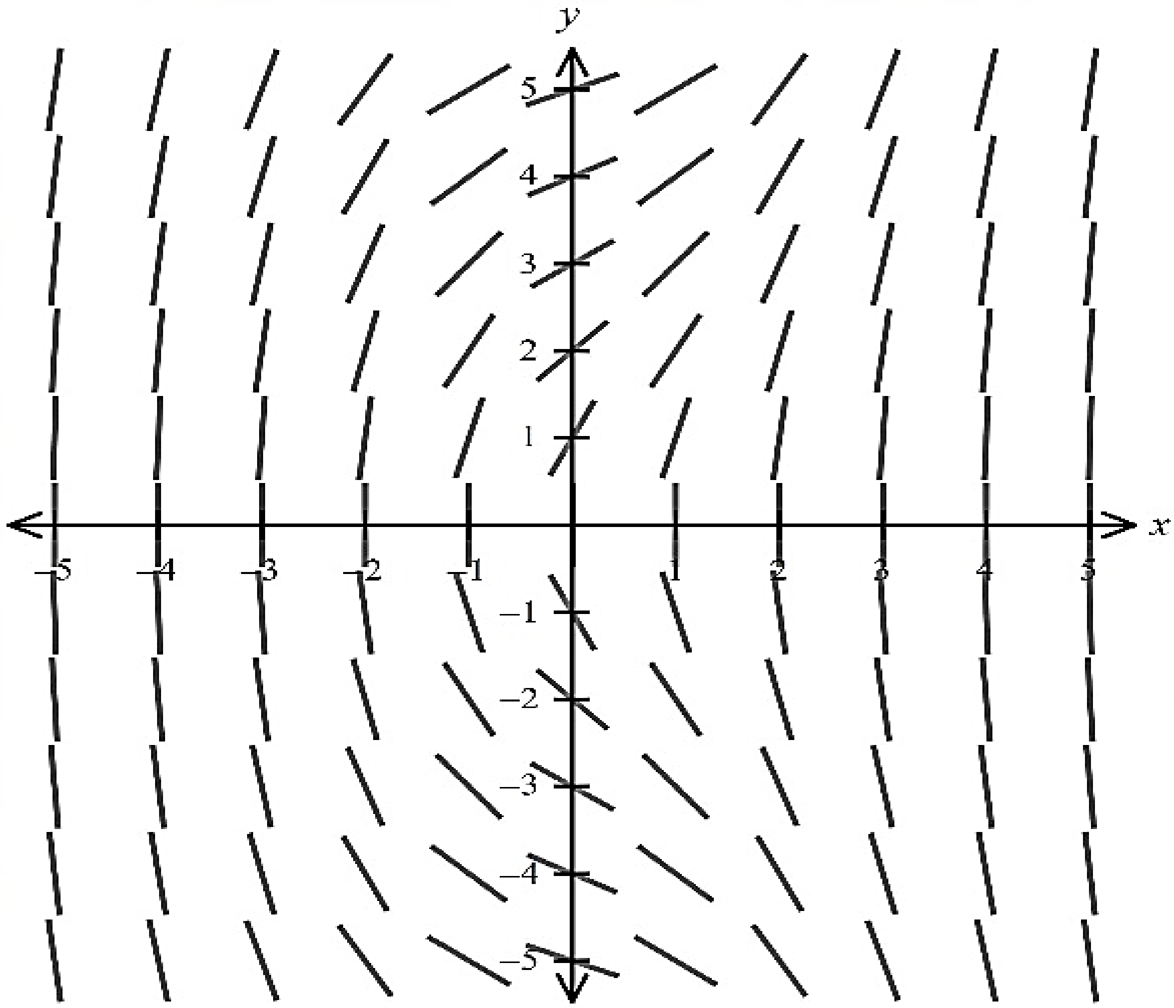
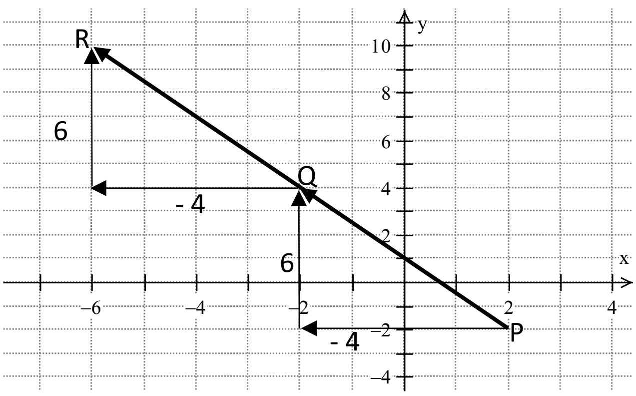


Mathematics Extension 1 Trial 2021 solutions

<table>
<tr><td colspan="3">Question Number</td></tr>
<tr><td>SUGGESTED SOLUTIONS</td><td>MARKS</td><td>MARKERS' COMMENT</td></tr>
<tr><td>1. $f(x) = cos^{-1}x$ has domain $-1 \le x \le 1$
So $f(x) = 2\,cos^{-1}2x$ has domain
$-1 \le 2x \le 1$ by dividing by 2, we get
$-\frac{1}{2} \le x \le \frac{1}{2}$ which can be
expressed as $\left[-\frac{1}{2}, \frac{1}{2}\right]$
Hence, the correct option is D.</td><td>1</td><td>D</td></tr>
<tr><td>2. Let $sinx + \sqrt{3}cosx = A\,sin(x+\theta)$
$sinx + \sqrt{3}cosx = A\,sinx.cos\theta + A\,cosx.sin\theta$
So $A\,cos\theta = 1$ (1) and
$A\,sin\theta = \sqrt{3}$ (2)
θ is in the first quadrant.
$\frac{A\,sin\theta}{A\,cos\theta} = \frac{\sqrt{3}}{1}$ (2) ÷ (1)
$tan\theta = \sqrt{3}$
$\theta = \frac{\pi}{3}$
Also $A^2sin^2\theta + A^2cos^2\theta = 1^2 + \sqrt{3}^2$ $(2)^2 + (1)^2$
$A^2(sin^2\theta + cos^2\theta) = 1 + 3$
$A^2 = 4$
$A = 2$
Hence, $sinx + \sqrt{3}cosx = 2\,sin\left(x + \frac{\pi}{3}\right)$
Hence, the correct option is A.</td><td>1</td><td>A</td></tr>
<tr><td>3. $T = A + Be^{-kt}$ satisfies $\frac{dT}{dt} = -k(T - A)$
$A = 15$ (surrounding air temperature)
When $t = 5$ $T = 75$
$75 = 15 + Be^{-5k}$
$Be^{-5k} = 60$ (1)
When $t = 8$ $T = 45$
$45 = 15 + Be^{-8k}$
$Be^{-8k} = 30$ (2)
Equation (1) divided by Equation (2)
$\frac{Be^{-5k}}{Be^{-8k}} = \frac{60}{30}$
$e^{3k} = 2$
$3k = \ln 2$
$k = \frac{\ln 2}{3}$
Hence, the correct option is D.</td><td>1</td><td>D</td></tr>
</table>

4. $$x = 2 + 4\cos\theta$$ $$y = 3 + 4\sin\theta$$ $$cos\theta = \frac{x-2}{4}$$ $$sin\theta = \frac{y-3}{4}$$ Now, $$sin^2\theta + cos^2\theta = 1$$ $\left(\frac{x-2}{4}\right)^{\mathbf{2}} + \left(\frac{y-3}{4}\right)^{\mathbf{2}}$=1 $$(x-2)^2 + (y-3)^2 = 16$$ Centre $(2, 3)$, radius 4. **Hence, the correct option is C.**	1	C
5. We can use the graphs provided to determine when the y value of the graph of $f(x) = \frac{1}{x}$ is greater than the y value on the graph of $g(x) = \frac{x}{4}$. We can see this occurs when $x < -2$ *or* $0 < x < 2$ **Hence, the correct option is C.**	1	C
6. This problem could be best solved graphically.  We are given that the coordinates of P are $(2, -2)$ The vector $\overrightarrow{PQ} = (-4, 6)$, so Q has coordinates $(-2, 4)$. Given $\overrightarrow{PR} = 2\overrightarrow{PQ}$, $\overrightarrow{QR} = \overrightarrow{PQ}$ so R will have coordinates $(-6, 10)$. Hence, the correct option is C.	1	C

7. The vector $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$ $= 6\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{i} + 3\underset{\sim}{j} + p\underset{\sim}{i} + (p+2)\underset{\sim}{j}$ $= (3+p)i + (10+p)j$ Now, as the magnitude of $\overrightarrow{OC}$ is 13 then $\sqrt{(3+p)^2 + (10+p)^2} = 13$ By squaring both sides, we get $(3+p)^2 + (10+p)^2 = 169$ $9 + 6p + p^2 + 100 + 20p + p^2 = 169$ $2p^2 + 26p - 60 = 0$ Dividing by 2, we get $p^2 + 13p - 30 = 0$ $(p+15)(p-2) = 0$ $p = -15 \ \ or \ \ p = 2$ As p is a positive constant then $p = 2$. **Hence, the correct option is B.**	**1**	**B**
8. The gradient at (-1,1) is $-\frac{1}{2(-1)} = \frac{1}{2}$, so options B and C are eliminated straight away as these have nega gradients. So, the only possible options are A and D. The gradient at (5,-5) is: $-\frac{-5}{2(5)} = \frac{1}{2}$. In option D, the gradient is negati at this point but the gradient should be positive. Therefore, A is the best option.	**1**	**A**
9. When selecting the five colours, we need always to select the sky - blue colour, as it is included in any group. This can be done in $^{1}C_{1} = 1$ way. Then we need to choose the other 4 colours and this can be done in $^{10}C_4$ ways. Hence, to select the colours there are $1 \times {^{10}C_4} = {^{10}C_4}$ ways. Now, to arrange the colours we start with the sky- blue star. There are 4 possible ways to place this colour and there 4! ways to arrange the other 4 colours. Hence, there are $^{10}C_4 \times 4 \times 4! = 4 \times {^{10}P_4}$ different ways to paint the five stars. **Hence, the correct option is B.** **Alternative method** The order of the colours of the stars is important so we are looking for permutations. If we assume the first star is painted sky blue, then	**1**	**B**

there are $^{10}P_4$ ways of painting the remaining 4 stars. If we assume the second star is painted sky blue, then there are $^{10}P_4$ ways of painting the remaining 4 stars. If we assume the fourth star is painted sky blue, then there are $^{10}P_4$ ways of painting the remaining 4 stars. If we assume the fifth star is painted sky blue, then there are $^{10}P_4$ ways of painting the remaining 4 stars. So, the total number of different permutations will be $4\times{}^{10}P_4$. Hence, the correct option is B.		
10. $\int_0^k \frac{1}{\sqrt{4-9x^2}}\,dx$ $\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}} = sin^{-1}(\frac{f(x)}{a}) + c$ ---from Reference Sheet $a = 2, f(x) = 3x$ and $f'(x) = 3$. Ideal format $=\frac{1}{3}\int_0^k \frac{3}{\sqrt{4-9x^2}}\,dx$ $=\frac{1}{3}(sin^{-1}(\frac{3x}{2})+c)$ $=\frac{1}{3}(sin^{-1}\left(\frac{3k}{2}\right) - sin^{-1}(0)$ $\frac{1}{3}sin^{-1}\left(\frac{3k}{2}\right) = \frac{\pi}{18}$ $\frac{3k}{2}=sin\frac{\pi}{6}$ $\frac{3k}{2} = \frac{1}{2}$ $3k = 1$ $k = \frac{1}{3}$	**1**	**D**

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11.a(i)

$$f(3)=0 \qquad f(-2)=0$$
$$27+9+3p+q=0 \qquad -8+4-2p+q=0$$
$$3p+q=-36 \qquad 2p-q=-4$$

Solve simultaneously by adding the 2 equations together

$$5p=-40$$
$$p=-8$$
$$q=-12$$

</td><td>1 mk-Correctly finds one linear equation.

½ mk- For each of the correct solutions.</td><td></td></tr>
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11. a(ii)

$$\begin{array}{r} x+2 \\ x^2-x-6\overline{)x^3+x^2-8x-12} \\ \underline{x^3-x^2-6x} \\ 2x^2-2x-12 \\ \underline{2x^2-2x-12} \\ 0 \end{array}$$

Therefore,

$f(x)=(x-3)(x+2)^2$.

</td><td>1 mk- For displaying the skill of long division.

1mk- correct solu</td><td></td></tr>
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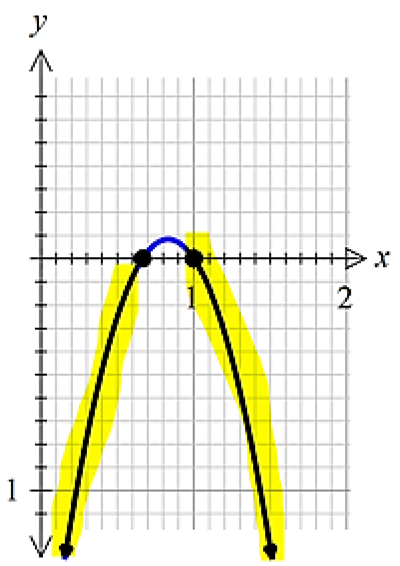
(b) $\dfrac{x}{x-1} \geq -2$

$$x \neq 1$$
$$x(x-1) \geq -2(x-1)^2$$
$$0 \geq -2(x-1)^2 - x(x-1)$$
$$0 \geq (x-1)[-2(x-1)-x]$$
$$0 \geq (x-1)(-3x+2)$$

$x \leq \dfrac{2}{3}$ or $x > 1$

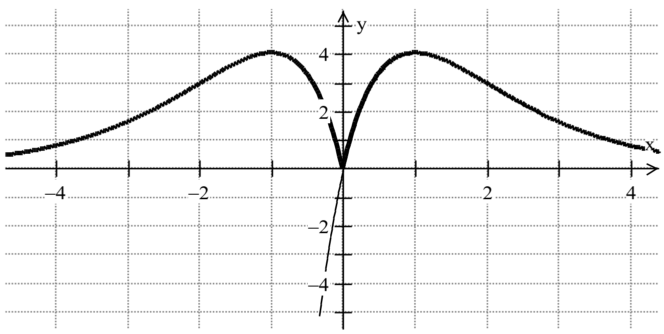


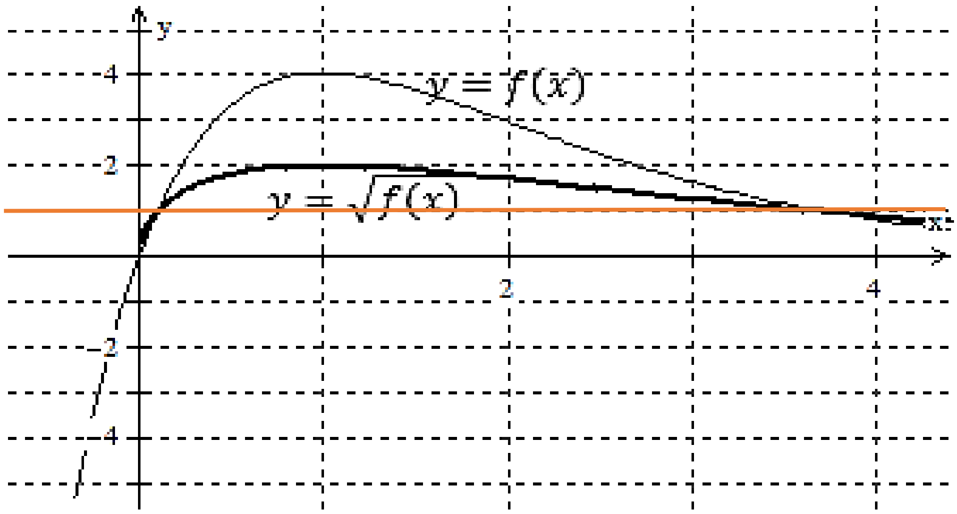
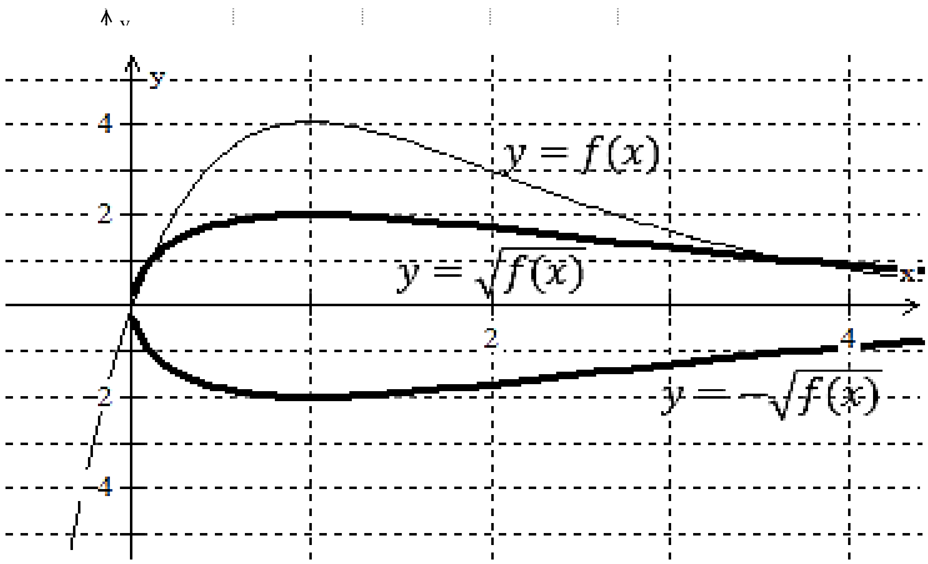
</td><td>1 mk-for multiplying by the square of $(x-1)$ and displaying the knowledge that x cannot be equal to 1, either through th solution or by explicitly stating it.

1mk- For correct factorisation.

1mk- for correct solution.</td><td></td></tr>
</table>

11c **Prove that the statement is true for the base case or initial case.** For $n = 1,\ 9^1 + 7 = 16$ which is divisible by 8. **Assume the statement is true for $n = k$.** That is, $9^k + 7 = 8p$, where p is a positive integer. That is, $9^k = 8p - 7$	1 mk-for the Initial case, assumption and the concluding stater	
Prove that the statement is true for $n = k + 1$. For $n = k + 1$ the statement is $9^{k+1} + 7 = 9 \times 9^k + 7$ $= 9 \times (8p - 7) + 7$ ---------From the assumption $= 72p - 63 + 7$ $= 72p - 56$ $= 8 \times (9p - 7)$ $= 8M$, where $M = 9p - 7$ is any integer. This is clearly divisible by 8.	1mk- for Using the assum appropriately. 1 mk- for appropriate expansion and simplification.	
Concluding statement Hence, if the statement is true for $n = k$ it is also true for $n = k + 1$. The statement was proved true for $n = 1$, hence by mathematical induction it is true for $n = 2$, $n = 3$ and so on. Hence, it is true for all values of $n \geq 1$.		
11d Considering the pairs of numbers which have a difference of 3: $\boxed{5,\ 8}$ $\boxed{6,\ 9}$ $\boxed{7,\ 10}$ Consider how many selections you can make, avoiding getting one of these pairs. If we selected 5, 6, 7 then we would not have any pairs with a difference of 3. But on our next selection, either 8, 9 or 10 we will have one of the pairs above. This is true for any selection where we avoid making the pairs for as long as possible. By the time you have selected 4 digits, you must have at least one pair.	1mk- for appropriate logical reasoning. 1mk- for correct answer.	

12a) Solve the differential equation $\frac{dy}{dx}=\frac{-x}{1+y^2}$, given that $y(-1)=1$ Express your answer in the form $ay^3+bx^2+cy+d=0$, where a,b,c,d are integers. $\frac{dy}{dx}=\frac{-x}{1+y^2}$ $dy\left(1+y^2\right)=-x\,dx$ $y+\frac{y^3}{3}=-\frac{x^2}{2}+C$ and $y(-1)=1$ $\boxed{\checkmark}$ $1+\frac{1^3}{3}=-\frac{(-1)^2}{2}+C\to C=\frac{11}{6}$ $\boxed{\checkmark}$ $y+\frac{y^3}{3}=-\frac{x^2}{2}+\frac{11}{6}\to 2y^3+3x^2+6y-11=0$ $\boxed{\checkmark}$	1mk- For correct expression, obtained at the stage indicated by the first tick. 1mk- For Calculating the correct value of C. 1mk-For Leaving the answer in the required form.	
12b) Use the substitution $u=3+e^x$ to find the exact value of $\int_0^{\ln 6}\frac{e^x}{\sqrt{3+e^x}}dx$ $u=3+e^x\to du=e^x dx$ and $u_1=3+e^{\ln 6}=9$ and $u_2=3+e^0=4$ $\boxed{\checkmark}$ $\int_0^{\ln 6}\frac{e^x}{\sqrt{3+e^x}}dx=\int_4^9\frac{du}{\sqrt{u}}$ ✓ $=2\left[\sqrt{u}\right]_4^9$ $=2\left(\sqrt{9}-\sqrt{4}\right)=2$ ✓	2mks-for displaying substantial knowledge and understanding of integration by substitution. 1mk- for the correct answer.	
12 c) i) The graph of $y=f(\lvert x\rvert)$ will be the same as the graph of $y=f(x)$ for $x\geq 0$ and will be the same as the graph of $y=f(-x)$ for $x\leq 0$. Graphically, the section of the graph where $x\geq 0$ will be reflected in the y axis, as seen below. 	1mk- For deleting part of the graph on the LHS of the y −axis. 1 mk- For reflecting the part of the graph on the RHS of the y − axis onto the y-axis.	
ii) The graph of $y^2=f(x)$ will be equivalent to the graphs of $y=\pm\sqrt{f(x)}$, or graphically the graph of $y=\sqrt{f(x)}$ and its reflection in the x axis.	½ mk- for the graph of $\sqrt{f(x)}$ to pass through the point with y=1.	

Where the graph of $y = f(x)$ is below the x axis, the graph of $y = \sqrt{f(x)}$ is not defined, so will not exist. The graph of $y = \sqrt{f(x)}$ will be 0 or 1 where the graph of $y = f(x)$ is 0 or 1 respectively. 	½ mk- for the shape of the graph of $\sqrt{f(x)}$for y-values between 0 to 1 as well as for y-values above 1.Also, delete part of the graph below the x −axis.	
Together the graphs of $y = \sqrt{f(x)}$ and $y = -\sqrt{f(x)}$ make up the graph of $y^2 = f(x)$. 	1 mk- for reflecting the graph of $\sqrt{f(x)}$ onto the x-axis.	
d) $\frac{dy}{dx} = \frac{4x^3}{sin^2y}$ so $sin^2y\ dy = 4x^3dx$ $\int sin^2y\ dy = \int 4x^3\,dx$ $\int \frac{1}{2}(1 - \cos 2y)\,dy = x^4 + c$ ---* $\frac{y}{2} - \frac{\sin 2y}{4} = x^4 + c$	1mk-for upto * 1mk- For final answer.	

13. a(i) Let $m = cosec^{-1}x$, then $x = cosec\, m$ and $\frac{1}{x} = \sin m$ so $m = sin^{-1}\frac{1}{x}$	1mk- for appropriate working.	
13a(ii) Hence $f(x) = cos^{-1}\frac{1}{x} + sin^{-1}\frac{1}{x}$ and $f'(x) = -\frac{-x^{-2}}{\sqrt{1-x^{-2}}} + \frac{-x^{-2}}{\sqrt{1-x^{-2}}}$ So $f'(x) = 0$	1mk -for differentiation of $cos^{-1}\frac{1}{x}$. 1mk -for differentiation of $sin^{-1}\frac{1}{x}$	
13. a(iii) As $f'(x) = 0$ then the continuous function $f(x)$ from part a, is constant, so by substituting any x value greater or equal to 1 we get this constant. Now, $f(x) = f(1) = \cos^{-1} 1 + \mathrm{cosec}^{-1} 1$ $= cos^{-1}1 + sin^{-1}1 = \frac{\pi}{2}$. Hence, $cos^{-1}\frac{1}{x} + cosec^{-1}x = \frac{\pi}{2}$ **Alternative Method** $cosec^{-1}x = sin^{-1}\frac{1}{x}$ Let $sin^{-1}\frac{1}{x} = m$, then $\sin m = \frac{1}{x}$ $\cos\left(\frac{\pi}{2} - m\right) = \frac{1}{x}$ $cos^{-1}\frac{1}{x} = \frac{\pi}{2} - m$ $cos^{-1}\frac{1}{x} = \frac{\pi}{2} - sin^{-1}\frac{1}{x}$ $cos^{-1}\frac{1}{x} = \frac{\pi}{2} - cosec^{-1}x$ Hence, $cos^{-1}\frac{1}{x} + cosec^{-1}x = \frac{\pi}{2}$.	1mk- for substitution of a number greater than 1 to obtain $\frac{\pi}{2}$.	

13b) Consider the points $P\ (a, 2a), Q\ (-\ a, 5a), R\ (3a, 4a)$ and $S\ (9a, 12a)$, where a is a positive real number. Given the length of the projection of $\overrightarrow{PQ}$ onto $\overrightarrow{RS}$ is 12, the value of a can be found as follows: $\overrightarrow{PQ} = -2a\,\underset{\sim}{i} + 3a\,\underset{\sim}{j}$ $\overrightarrow{RS} = 6a\,\underset{\sim}{i} + 8a\,\underset{\sim}{j}$ The projection of $\overrightarrow{PQ}$ onto $\overrightarrow{RS}$ equals $= \frac{\overrightarrow{PQ}.\ \overrightarrow{RS}}{\lvert\overrightarrow{RS}\rvert^2}\overrightarrow{RS}$ $= \frac{-12\,a^2 + 24\,a^2}{36\,a^2 + 64\,a^2} \times (\,6a\,\underset{\sim}{i} + 8a\,\underset{\sim}{j}\,)$ $= \frac{12}{100}(\,6a\,\underset{\sim}{i} + 8a\,\underset{\sim}{j}\,)$ $= \frac{3}{25}(6a\,\underset{\sim}{i} + 8a\,\underset{\sim}{j})$ $= \frac{18}{25}a\,\underset{\sim}{i} + \frac{24}{25}a\,\underset{\sim}{j}$ The length of this vector is $\sqrt{\left(\frac{18}{25}\right)^2 a^2 + \left(\frac{24}{25}\right)^2 a^2} = 12$ $\frac{a}{25}\sqrt{18^2 + 24^2} = 12$ $\frac{30a}{25} = 12$ $a = 12 \times \frac{25}{30}$ Hence, $a = 10$.	1 mk- for finding $\overrightarrow{PQ} = -2a\,\underset{\sim}{i} + 3a\,\underset{\sim}{j}$ and $\overrightarrow{RS} = 6a\,\underset{\sim}{i} + 8a\,\underset{\sim}{j}$ 1mk- for finding the projection vector. 1mk- for finding the value of a.	

13 c(i) $y=2\ln(x-1)$ $\frac{1}{2}y=\ln(x-1)$ $x-1=e^{\frac{1}{2}y}$ $x=e^{\frac{1}{2}y}+1$	1mk- for obtaining $x=e^{\frac{1}{2}y}+1$	
$V=\pi\int x^2dy$ $=\pi\int_0^p\left(e^y+2e^{\frac{1}{2}y}+1\right)dy$ $=\pi\left[e^y+4e^{\frac{1}{2}y}+y\right]_0^p$ $=\pi\left[\left(e^p+4e^{\frac{1}{2}p}+p\right)-(1+4+0)\right]$ $=\pi\left(e^p+4e^{\frac{1}{2}p}+p-5\right)$	1mk- for Correct substitution into $V=\pi\int x^2dy$ 1mk- for correct integration and appropriate substitution.	
(ii) $V=\pi\left(e^p+4e^{\frac{1}{2}p}+p-5\right)$ $\frac{dV}{dp}=\pi\left(e^p+2e^{\frac{1}{2}p}+1\right)$ When $p=4$, $\frac{dV}{dp}=\pi\left(e^4+2e^2+1\right)$ Using the chain rule, $\frac{dV}{dt}=\frac{dp}{dt}\times\frac{dV}{dp}$ $\frac{dV}{dt}=0.2\times\pi\left(e^4+2e^2+1\right)$ $\frac{dV}{dt}=44\text{ cm}^3\text{s}^{-1}$ correct to 2sf.	1mk-For correct differentiation of V with respect to p. 1mk-For correct substitution into the chain rule expression to obtain the value for $\frac{dV}{dt}$:	

14.a) (i) Start with the RHS and show that it equals the LHS. $\text{RHS} = \frac{1}{50}\left(\frac{(50-A)+A}{A(50-A)}\right)$ $= \frac{1}{A(50-A)}$ $= \text{LHS}$	1 mk- for appropriate simplification.	
(ii) This is a differential equation of the form $\frac{dA}{dt} = g(A)$. Attempt to separate variables and integrate both sides. $\int 1\,dt = \int \frac{25}{A(50-A)}dA$ $t = \frac{1}{2}\int\left(\frac{1}{A} + \frac{1}{50-A}\right)dA$ (using the part (i) result) $= \frac{1}{2}(\ln\lvert A\rvert - \ln\lvert 50-A\rvert) + c$ $= \frac{1}{2}\ln\left\lvert\frac{A}{50-A}\right\rvert + c$ $\mathbf{2t - 2c = \left\lvert\frac{A}{50-A}\right\rvert}$ $\boldsymbol{e^{2t-2c} = \left\lvert\frac{A}{50-A}\right\rvert}$ $\boldsymbol{e^{2t}.e^{-2c} = \left\lvert\frac{A}{50-A}\right\rvert}$ Let $A_0 = e^{-2c}$ $\therefore \left\lvert\frac{A}{50-A}\right\rvert = A_0e^{2t}$, where $\mathrm{A_0}$ is any positive constant. $\frac{A}{50-A} = \pm A_0e^{2t}$ $\therefore \frac{A}{50-A} = A_0e^{2t}$, where $\mathrm{A_0}$ is any constant When $t = 0$, $A = \frac{1}{2}$ and so $A_0 = \frac{1}{99}$.	1 mk- for $\int 1\,dt = \int \frac{25}{A(50-A)}dA$ 1 mk-for $\frac{A}{50-A} = A_0e^{2t}$, where A_0 is any constant.	

$$e^{2t} = \frac{99A}{50-A}$$ $$99Ae^{-2t} = 50 - A$$ $$A(1+99e^{-2t}) = 50$$ So $A = \frac{50}{1+99e^{-2t}}$.	1 mk- for working out the value of A_0 and Then appropriate simplification. to obtain $A = \frac{50}{1+99e^{-2t}}$	
As $t \to \infty$, $1 + 99e^{-2t} \to 1$ and so $A \to \frac{50}{1} = 50$. The limiting area of the bacteria colony is 50 cm^2.	1mk- For the correct limiting value.	
b) Let $u = \sin x$ $$\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx = \int_0^1 u^2 \, du$$ $$= \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$	1mk- for appropriate substitution. 1mk- for the final answer.	
c(i) Let $u = x$ $$u' = 1$$ $$v = \cos^{-1} x$$ $$v' = -\frac{1}{\sqrt{1-x^2}}$$ $$\frac{d}{dx} x \cos^{-1} x = vu' + uv'$$ $$= \cos^{-1} x - \frac{x}{\sqrt{1-x^2}}$$	1mk-for Correct usage of product rule with minor errors.	

c (ii)

$$\int \cos^{-1}x \; - \frac{x}{\sqrt{1-x^2}}\,dx = x\,\cos^{-1}x$$

$$\int \cos^{-1}x\,dx = x\,\cos^{-1}x + \int\left(\frac{x}{\sqrt{1-x^2}}\right)\,dx$$

1mk-for the correct expression for $\int \cos^{-1}x\,dx$

Let $u = 1 - x^2$

$$du = -2x\,dx$$

$$\int\left(\frac{x}{\sqrt{1-x^2}}\right)dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,\mathrm{du}$$

$$= -\frac{1}{2}\left(2u^{\frac{1}{2}}\right)$$

$$= -\sqrt{u}$$

$$= -\sqrt{1-x^2}$$

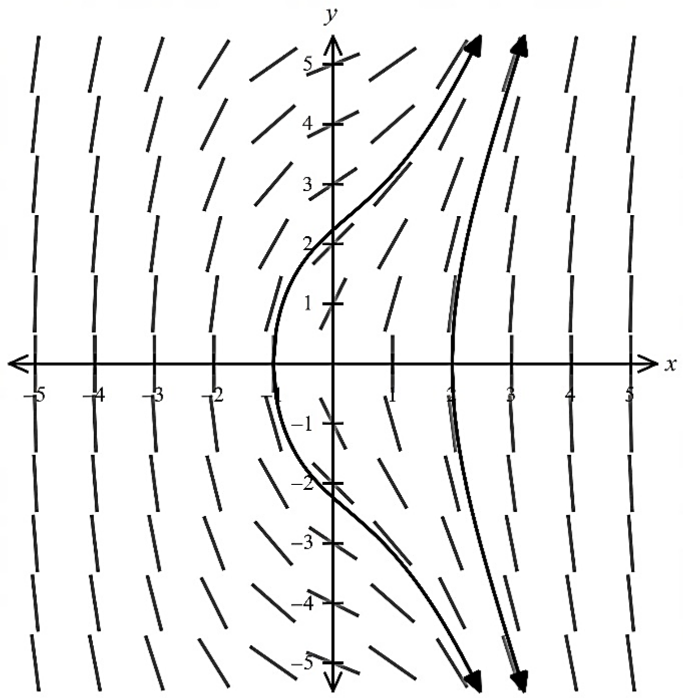
1mk- for correct integration of $\int\left(\frac{x}{\sqrt{1-x^2}}\right)dx$

$$\therefore \int \cos^{-1}x\,dx = x\,\cos^{-1}x - \sqrt{1-x^2} + C$$

d)

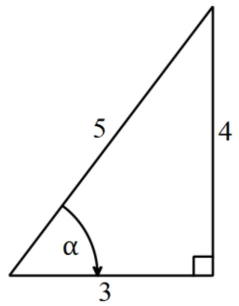
$\frac{dy}{dx} = \frac{3x^2+4}{2y}$ Note when $y = 0$, $\frac{dy}{dx}$ is undefined, so a vertical gradient on x-axis at x=-1 and at x= 2. . Follow slope lines to get shape of curve from there.

If equation in (i) doesn't have $\pm$ then only section above x axis is the graph. (Don't penalise if student shows the whole grasph)

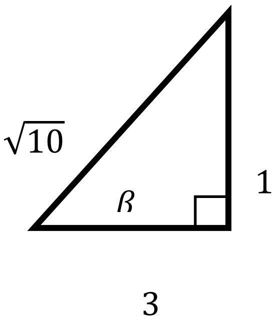


2 marks for a diagram with 2 correct graphs

1 mark for one correct graph or two incorrect graphs with the same minor error.

15. When $y=0,\ t\left(-\frac{1}{2}gt+V\sin\alpha\right)=0$ $t=0$, or $-\frac{1}{2}gt+V\sin\alpha=0$ $\frac{1}{2}gt=V\sin\alpha$ $t=\frac{2V\sin\alpha}{g}$	1mk-for making $y=0$ and then solving for t.	
(ii) $OA=Vt\cos\alpha$ $=V\left(\frac{2V\sin\alpha}{g}\right)\cos\alpha$ $=\frac{V^2\sin 2\alpha}{g}$ **For Particle P:** If $\tan\alpha=\frac{4}{3}$, $\sin\alpha=\frac{4}{5}$ $\cos\alpha=\frac{3}{5}$  Therefore, $OA=\frac{V^2 2\sin\alpha\cos\alpha}{g}$ $=\frac{2V^2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)}{g}$ $=\frac{24V^2}{25g}$	1mk-for substituting the expression for t in (i) into the expression for x.	

Time of flight (when $y=0$) $t=\dfrac{2V\sin\alpha}{g}$ $=\dfrac{2V\left(\frac{4}{5}\right)}{g}$ $=\dfrac{8V}{5g}$ Highest point of flight (when $\dot{y}=0$) and the time is ½ of what is needed for P to land at A. $t=\dfrac{1}{2}\times\dfrac{8V}{5g}$ $=\dfrac{4V}{5g}$ Equations of motion for particle Q are $x=UT\cos\beta$ $y=-\dfrac{1}{2}gT^2+UT\sin\beta$ where $T=\dfrac{4V}{5g}$ Substituting into equation for x $\dfrac{24V^2}{25g}=U\left(\dfrac{4V}{5g}\right)\cos\beta$ $24V=20U\cos\beta$ $6V=5U\cos\beta$ Substituting into equation for y $0=-\dfrac{1}{2}g\left(\dfrac{4V}{5g}\right)^2+U\left(\dfrac{4V}{5g}\right)\sin\beta$ $\dfrac{16V^2}{25g}=2U\left(\dfrac{4V}{5g}\right)\sin\beta$ $16V=40U\sin\beta$ $2V=5U\sin\beta$ Solve simultaneously, $\dfrac{5U\sin\beta}{5U\cos\beta}=\dfrac{2V}{6V}$ $\tan\beta=\dfrac{1}{3}$	1mk 1mk 1mk	



$$\therefore \sin\beta = \frac{1}{\sqrt{10}}$$

1mk

$$2V = 5U\sin\beta$$

$$2V = 5U \times \frac{1}{\sqrt{10}}$$

$$U = \frac{2\sqrt{10}V}{5}$$

b)

i) $cos4\theta = 2cos^2 2\theta - 1$

$cos4\theta = 2(2cos^2\theta - 1)^2 - 1$

$cos4\theta = 2(4cos^4\theta - 4cos^2\theta + 1) - 1$

$cos4\theta = 8cos^4\theta - 8cos^2\theta + 2 - 1$

$cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$

$0 = 8cos^4\theta - 8cos^2\theta + 1 - cos4\theta$

1mk- for appropriate substitution of $\cos 2\theta = 2cos^2\theta - 1$

1mk- for correct expansion and simplification.

ii) $cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$

$2cos4\theta = 16cos^4\theta - 16cos^2\theta + 2$

By substituting

$16cos^4\theta - 16cos^2\theta + 2 - \sqrt{2} = 0$

from above, we get

$2cos4\theta - \sqrt{2} = 0$

$2cos4\theta = \sqrt{2}$

$cos4\theta = \frac{\sqrt{2}}{2}$

1mk- for appropriate substitution of part(i).

1mk-for appropriate simplification.

(iii) From part ii) the solution to the equation $16cos^4\theta - 16cos^2\theta + 2 - \sqrt{2} = 0$ can be found from $\cos 4\theta = \frac{\sqrt{2}}{2}$ So $4\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$ .... $\therefore\ \theta = \frac{\pi}{16}, \frac{7\pi}{16}, \frac{9\pi}{16}, \frac{15\pi}{16}$ .... are solutions to the equation. But $16cos^4\theta - 16cos^2\theta + 2 - \sqrt{2} = 0$ can be reduced to the quadratic $16m^2 - 16m + 2 - \sqrt{2} = 0$ where $m = cos^2\theta$ As $\theta = \frac{\pi}{16}, \frac{7\pi}{16}, \frac{9\pi}{16}, \frac{15\pi}{16}$ .... are solutions to the initial trig equation, $cos^2\frac{\pi}{16}$ $and\ cos^2\frac{7\pi}{16}$ will be roots of this new equation. The product of the roots of a quadratic equation is equal to the constant term, divided by the coefficient of m^2, Hence, $cos^2\frac{\pi}{16}.cos^2\frac{7\pi}{16} = \frac{2-\sqrt{2}}{16}$	1mk-for solving for θ. 1mk- for appropriate justification using the product of roots of a quadratic equation.	